

ON MEAGER FUNCTION SPACES, NETWORK CHARACTER AND MEAGER CONVERGENCE IN TOPOLOGICAL SPACES

TARAS BANAKH, VOLODYMYR MYKHAYLYUK, LYUBOMYR ZDOMSKYY

ABSTRACT. For a non-isolated point x of a topological space X let $\text{nw}_X(x)$ be the smallest cardinality of a family \mathcal{N} of infinite subsets of X such that each neighborhood $O(x) \subset X$ of x contains a set $N \in \mathcal{N}$. We prove that

- each infinite compact Hausdorff space X contains a non-isolated point x with $\text{nw}_X(x) = \aleph_0$;
- for each point $x \in X$ with $\text{nw}_X(x) = \aleph_0$ there is an injective sequence $(x_n)_{n \in \omega}$ in X that \mathcal{F} -converges to x for some meager filter \mathcal{F} on ω ;
- if a functionally Hausdorff space X contains an \mathcal{F} -convergent injective sequence for some meager filter \mathcal{F} , then for every path-connected space Y that contains two non-empty open sets with disjoint closures, the function space $C_p(X, Y)$ is meager.

Also we investigate properties of filters \mathcal{F} admitting an injective \mathcal{F} -convergent sequence in $\beta\omega$.

This paper was motivated by a question of the second author who asked if the function space $C_p(\omega^*, 2)$ is meager. Here $\omega^* = \beta\omega \setminus \omega$ is the remainder of the Stone-Čech compactification of the discrete space of finite ordinals ω and $2 = \{0, 1\}$ is the doubleton endowed with the discrete topology. According to Theorem 4.1 of [13] this question is tightly connected with the so-called meager convergence of sequences in ω^* .

A filter \mathcal{F} on ω is *meager* if it is meager (i.e., of the first Baire category) in the power-set $\mathcal{P}(\omega) = 2^\omega$ endowed with the usual compact metrizable topology. By the Talagrand characterization [18], a free filter \mathcal{F} on ω is meager if and only if $\xi(\mathcal{F}) = \mathfrak{F}r$ for some finite-to-one function $\xi : \omega \rightarrow \omega$. A function $\xi : \omega \rightarrow \omega$ is *finite-to-one* if for each point $y \in \omega$ the preimage $\xi^{-1}(y)$ is finite and non-empty. A filter \mathcal{F} on ω is defined to be ξ -meager for a surjective function $\xi : \omega \rightarrow \omega$ if $\xi(\mathcal{F}) = \mathfrak{F}r$.

We shall say that for a filter \mathcal{F} on ω , a sequence $(x_n)_{n \in \omega}$ of points of a topological space X \mathcal{F} -converges to a point $x_\infty \in X$ if for each neighborhood $O(x_\infty) \subseteq X$ of x_∞ the set $\{n \in \omega : x_n \in O(x_\infty)\}$ belongs to the filter \mathcal{F} . Observe that the usual convergence of sequences coincides with the $\mathfrak{F}r$ -convergence for the Fréchet filter $\mathfrak{F}r = \{A \subseteq \omega : \omega \setminus A \text{ is finite}\}$ that consists of all cofinite subsets of ω . The filter convergence of sequences has been actively studied both in Analysis [1], [4] and Topology [5]. A sequence $(x_n)_{n \in \omega}$ will be called *meager-convergent* if it is \mathcal{F} -convergent for some meager filter \mathcal{F} on ω . A sequence $(x_n)_{n \in \omega}$ is called *injective* if $x_n \neq x_m$ for all $n \neq m$.

We shall prove that for a zero-dimensional Hausdorff space X the function space $C_p(X, 2)$ is meager if X contains an injective meager-convergent sequence. We recall that a topological space X is *functionally Hausdorff* if for any distinct points $x, y \in X$ there is a continuous function $\lambda : X \rightarrow \mathbb{I}$ such that $\lambda(x) \neq \lambda(y)$. Here $\mathbb{I} = [0, 1]$ is the unit interval. For topological spaces X, Y by $C_p(X, Y)$ we denote the space of continuous functions endowed with the topology of pointwise convergence.

Theorem 1. *Let X be a functionally Hausdorff space and Y be a topological space that contains two open non-empty subsets with disjoint closures. Assume that X is zero-dimensional or Y is path-connected. If X contains an injective meager-convergent sequence, then the function space $C_p(X, Y)$ is meager.*

Proof. Let $(x_n)_{n \in \omega}$ be a sequence in X that \mathcal{F} -converges to $x_\infty \in X$ for some meager filter \mathcal{F} in ω . Then there is a finite-to-one surjection $\xi : \omega \rightarrow \omega$ such that $\xi(\mathcal{F}) = \mathfrak{F}r$. By our assumption, Y contains two non-empty open subsets W_0, W_1 with disjoint closures.

For every $n \in \omega$ consider the subset

$$C_n = \{f \in C_p(X, Y) : \forall i \in \{0, 1\} (f(x_\infty) \notin \overline{W_i} \Rightarrow \forall m \geq n \exists k \in \xi^{-1}(m) (f(x_k) \notin \overline{W_i}))\}.$$

2000 *Mathematics Subject Classification.* Primary: 54A20, 54C35; Secondary: 54E52.

Key words and phrases. network character, meager convergent sequence, meager filter, meager space, function space.

The third author acknowledges the support of FWF grant P19898-N18.

The meager property of $C_p(X, Y)$ will follow as soon as we check that $C_p(X, Y) = \bigcup_{n \in \omega} \mathcal{C}_n$ and each set \mathcal{C}_n is nowhere dense in $C_p(X, Y)$.

To show that $C_p(X, Y) = \bigcup_{n \in \omega} \mathcal{C}_n$, fix any continuous function $f \in C_p(X, Y)$. Since $Y = (Y \setminus \overline{W}_0) \cup (Y \setminus \overline{W}_1)$, there is $i \in \{0, 1\}$ such that $f(x_\infty) \notin \overline{W}_i$. Since (x_n) is \mathcal{F} -convergent to x_∞ and $f^{-1}(Y \setminus \overline{W}_i)$ is an open neighborhood of x_∞ , the set $F = \{n \in \omega : f(x_n) \notin \overline{W}_i\}$ belongs to the filter \mathcal{F} and thus the image $\xi(F)$, being cofinite in ω , contains the set $\{m \in \omega : m \geq n\}$ for some $n \in \omega$. Then $f \in \mathcal{C}_n$ by the definition of the set \mathcal{C}_n .

Next, we show that each set \mathcal{C}_n is nowhere dense in $C_p(X, Y)$. Fix any non-empty open set $\mathcal{U} \subseteq C_p(X, Y)$. Without loss of generality, \mathcal{U} is a basic open set of the following form:

$$\mathcal{U} = \{f \in C_p(X, Y) : \forall z \in Z \ f(z) \in U_z\}$$

for some finite set $Z \subseteq X$ and non-empty open sets $U_z \subseteq Y$, $z \in Z$. We can additionally assume that $x_\infty \in Z$. We need to find a non-empty open set $\mathcal{V} \subseteq C_p(X, Y)$ such that $\mathcal{V} \subseteq \mathcal{U} \setminus \mathcal{C}_n$. If $\mathcal{U} \cap \mathcal{C}_n$ is empty, then put $\mathcal{V} = \mathcal{U}$. So we assume that $\mathcal{U} \cap \mathcal{C}_n$ contains some function f_0 . For this function we can find $i \in \{0, 1\}$ such that $f_0(x_\infty) \notin \overline{W}_i$. Since $f_0(x_\infty) \in U_{x_\infty}$, we lose no generality assuming that $U_{x_\infty} \subseteq Y \setminus \overline{W}_i$.

Since the sequence $(x_n)_{n \in \omega}$ is injective, we can find $m \geq n$ such that the set $X_m = \{x_k : k \in \xi^{-1}(m)\}$ does not intersect the finite set Z . Choose any function $g : Z \cup X_m \rightarrow Y$ such that $g(z) = f_0(z)$ for all $z \in Z$ and $g(x) \in W_{1-i}$ for all $x \in X_m$.

We claim that the function g has a continuous extension $\bar{g} : X \rightarrow Y$. By our assumption, X is zero-dimensional or Y path-connected. In the first case we can find a retraction $r : X \rightarrow Z \cup X_m$ and put $\bar{g} = g \circ r$. If Y is path-connected, then take any injective function $\phi : g(Z \cup X_m) \rightarrow \mathbb{I}$ and extend the function $\phi \circ g : Z \cup X_m \rightarrow \mathbb{I}$ to a continuous map $\lambda : X \rightarrow \mathbb{I}$ using the functional Hausdorff property of X . Since Y is path-connected, the map $\phi^{-1} : (\phi \circ g)(Z \cup X_m) \rightarrow Y$ extends to a continuous map $\psi : \mathbb{I} \rightarrow Y$. Then the continuous map $\bar{g} = \psi \circ \lambda : X \rightarrow Y$ is a required continuous extension of g .

In both cases the set

$$\mathcal{V} = \{f \in C_p(X, Y) : \forall z \in Z \ f(z) \in U_z, \text{ and } \forall x \in X_m \ f(x) \in W_{1-i}\}$$

is an open neighborhood of \bar{g} that lies in $\mathcal{U} \setminus \mathcal{C}_n$, witnessing that the set \mathcal{C}_n is nowhere dense in $C_p(X, Y)$. \square

In light of Theorem 1 it is important to detect topological spaces that contain injective meager-convergent sequences. This will be done for spaces containing points with countable network character.

A family \mathcal{N} of subsets of a topological space X is called a π -network at a point $x \in X$ if each neighborhood $O(x) \subset X$ of x contains some set $N \in \mathcal{N}$. If each set $N \in \mathcal{N}$ is infinite, then \mathcal{N} will be called an i -network at x . An i -network at x exists if and only if each neighborhood of x in X is infinite. In this case let $\text{nw}_\chi(x; X)$ denote the smallest cardinality $|\mathcal{N}|$ of an i -network \mathcal{N} at x . If some neighborhood of x in X is finite, then let $\text{nw}_\chi(x; X) = 1$. If the space X is clear from the context, then we write $\text{nw}_\chi(x)$ instead of $\text{nw}_\chi(x; X)$ and call this cardinal the *network character* of x in X . If X is a T_1 -space, then $\text{nw}_\chi(x) \geq \aleph_0$ if and only if the point x is not isolated in X . The cardinal $\text{hnw}_\chi(x) = \sup\{\text{nw}_\chi(x; A) : x \in A \subset X\}$ is called the *hereditary network character* at x . Points $x \in X$ with $\text{hnw}_\chi(x) \leq \aleph_0$ are called *Pytkeev points*, see [11].

Theorem 2. *If some point x of a topological space X has $\text{nw}_\chi(x) = \aleph_0$, then for each finite-to-one function $\xi : \omega \rightarrow \omega$ with $\lim_{n \rightarrow \infty} |\xi^{-1}(n)| = \infty$ there is an injective sequence $(x_n)_{n \in \omega}$ in X that \mathcal{F} -converges to x for some ξ -meager filter \mathcal{F} .*

Proof. Let $(N_i)_{i \in \omega}$ be a countable i -network at x . Since each set N_i is infinite, we can choose an injective sequence $(x_k)_{k \in \omega}$ in X such that for every $n \in \omega$ and $0 \leq i < |\xi^{-1}(n)|$ the set N_i meets the set $\{x_k : k \in \xi^{-1}(n)\}$.

It is clear that the sequence $(x_n)_{n \in \omega}$ \mathcal{F} -converges to x for the filter

$$\mathcal{F} = \{\{n \in \omega : x_n \in O(x)\} : O(x) \text{ is a neighborhood of } x \text{ in } X\}.$$

It remains to check that the filter \mathcal{F} is ξ -meager. Given any neighborhood $O(x) \subset X$ of x we need to find $n \in \omega$ such that for every $m \geq n$ there is $k \in \xi^{-1}(m)$ with $x_k \in O(x)$. Since $(N_i)_{i \in \omega}$ is a network at x , there is $i \in \omega$ such that $N_i \subset O(x)$. Taking into account that $\lim_{n \rightarrow \infty} |\xi^{-1}(n)| = \infty$, find $n \in \omega$ such that $|\xi^{-1}(m)| > i$ for all $m \geq n$. Now the choice of the sequence (x_k) guarantees that for every $m \geq n$ there is $k \in \xi^{-1}(m)$ with $x_k \in N_i \subset O(x)$. \square

In light of Theorem 2 it is important to detect points x with countable network character $\text{nw}_\chi(x)$. Let us recall that the *character* $\chi(x)$ (resp. the π -character $\pi\chi(x)$) of a point x in a topological space X is equal to

the smallest cardinality of a neighborhood base (resp. a π -base) at x . A π -base at x is any π -network at x consisting of non-empty open subsets of X . These definitions imply the following simple:

Proposition 1. *For any non-isolated point x of a T_1 -space X ,*

- (1) $\text{nw}_X(x) \leq \chi(x)$;
- (2) $\text{nw}_X(x) \leq \pi\chi(x)$ provided that x has a neighborhood containing no isolated point of X ;
- (3) $\text{nw}_X(x) = \aleph_0$ if x is the limit of an injective \mathfrak{F} -convergent sequence in X .

The following simple example shows that the usual convergence of the injective sequence in Proposition 1(3) cannot be replaced by the meager convergence. It also shows that Theorem 2 cannot be reversed.

Example 1. Let \mathcal{F} be the meager filter on ω consisting of the sets $F \subset \omega$ such that

$$\lim_{n \rightarrow \infty} \frac{|F \cap [2^n, 2^{n+1})|}{2^n} = 1.$$

On the space $X = \omega \cup \{\infty\}$ consider the topology in which all points $n \in \omega$ are isolated while the sets $F \cup \{\infty\}$, $F \in \mathcal{F}$, are neighborhoods of ∞ . It is clear that the sequence $x_n = n$, $n \in \omega$, \mathcal{F} -converges to ∞ in X . On the other hand, a simple diagonal argument shows that $\text{nw}_X(\infty; X) > \aleph_0$.

Theorem 3. *Each infinite compact Hausdorff space X contains a point $x \in X$ with $\text{nw}_X(x) = \aleph_0$.*

Proof. Theorem trivially holds if X contains a non-trivial convergent sequence. So we assume that X contains no non-trivial convergent sequence. Then X contains a closed subset $C \subset X$ that admits a continuous map $g : C \rightarrow \mathbb{I}$ onto the unit interval $\mathbb{I} = [0, 1]$, see [7, p.172]. Replacing C by a smaller subset, we can assume that the map $g : C \rightarrow \mathbb{I}$ is irreducible, which means that $g(C') \neq \mathbb{I}$ for any proper closed subset $C' \subset C$. Fix any countable base \mathcal{B} of the topology of \mathbb{I} . The irreducibility of the map $g : C \rightarrow \mathbb{I}$ implies that the space C has no isolated points. Also the irreducibility of g implies that the countable family $\mathcal{N} = \{g^{-1}(U) : U \in \mathcal{B}\}$ of open infinite subsets of C is an i -network at each point $x \in C$. Consequently, $\text{nw}_X(x) = \aleph_0$ for each point $x \in C$. \square

Theorems 1—3 imply:

Corollary 1. *For each infinite zero-dimensional compact Hausdorff space X and each topological space Y containing two non-empty open sets with disjoint closures the function space $C_p(X, Y)$ is meager. In particular, the function space $C_p(\omega^*, 2)$ is meager.*

Also Theorems 2 and 3 imply

Corollary 2. *Let $\xi : \omega \rightarrow \omega$ be a finite-to-one function with $\lim_{n \rightarrow \infty} |\xi^{-1}(n)| = \infty$. Each infinite compact Hausdorff space X contains an injective \mathcal{F} -convergent sequence for some ξ -meager filter \mathcal{F} on ω .*

In fact, the condition $\lim_{n \rightarrow \infty} |\xi^{-1}(n)| = \infty$ in Corollary 2 can not be weakened.

Let us recall that an infinite subset A is called a *pseudointersection* of a family of sets \mathcal{F} if $A \subseteq^* F$ for all $F \in \mathcal{F}$ where $A \subseteq^* F$ means that $A \setminus F$ is finite. If a sequence $(x_n)_{n \in \omega}$ in a topological space \mathcal{F} -converges to a point x_∞ for some filter \mathcal{F} with infinite pseudointersection $A \subseteq \omega$ then the subsequence $(x_k)_{k \in A}$ converges to x_∞ in the standard sense.

Lemma 1. *Let I be a countable set and $C = \bigcup_{i \in I} C_i$, where the sets C_i are nonempty and mutually disjoint, and $\sup_{i \in I} |C_i| < \omega$. If \mathcal{H} is a filter on C all of whose elements intersect all but finitely many C_i 's, then \mathcal{H} has an infinite pseudointersection.*

Proof. The proposition will be proved by induction on $n = \sup_{i \in I} |C_i|$. If $n = 1$ there is nothing to prove. Suppose that it is true for all $k < n$ and let I , $\{C_i : i \in I\}$, \mathcal{H} be as above with $\max\{|C_i| : i \in I\} = n$. If for every $H \in \mathcal{H}$ the set $\{i \in I : |C_i \cap H| < n\}$ is finite, then C itself is a pseudointersection of \mathcal{H} . So suppose that $J = \{i \in I : |C_i \cap H_0| < n\}$ is infinite for some $H_0 \in \mathcal{H}$. In this case we may use our inductive hypothesis for J , $\{C_i \cap H_0 : i \in J\}$, $\mathcal{G} = \mathcal{H} \upharpoonright (\bigcup_{i \in J} C_i \cap H_0)$, and $n - 1$. Thus \mathcal{G} has an infinite pseudointersection, and hence so does \mathcal{H} . \square

Proposition 2. *If \mathcal{F} is a ξ -meager filter on ω for some surjective function $\xi : \omega \rightarrow \omega$ with $\lim_{n \rightarrow \infty} |\xi^{-1}(n)| < \infty$, then any sequence $(x_n)_{n \in \omega}$ in a topological space X that \mathcal{F} -converges to a point $x_\infty \in X$ contains a subsequence $(x_{n_k})_{k \in \omega}$ that converges to x_∞ .*

Proof. Choose infinite set $I \subseteq \omega$ such that $\sup_{i \in I} |\xi^{-1}(i)| < \omega$. Let $C_i = \xi^{-1}(i)$ for every $i \in I$, $C = \bigcup_{i \in I} C_i$ and $\mathcal{H} = \{F \cap C : F \in \mathcal{F}\}$. According to Lemma 1 there exists an infinite set $D \subseteq C$ such that $D \subseteq^* H$ for every $H \in \mathcal{H}$. Then the subsequence $(x_i)_{i \in D}$ converges to x_∞ . \square

Now let us compare two facts:

- (1) the compact Hausdorff space $\beta\omega$ contains no injective $\mathfrak{F}r$ -convergent sequences;
- (2) each infinite compact Hausdorff space X contains an injective \mathcal{F} -convergent sequence for some meager filter \mathcal{F} .

These two facts suggest a problem of finding the borderline between filters \mathcal{F} that admit an injective \mathcal{F} -convergent sequence in $\beta\omega$ and filters that admit no such sequences. We hope that this borderline passes near analytic filters. Let us recall the definitions of some properties of filters.

A filter \mathcal{F} is *analytic* (resp. an F_σ -filter, $F_{\sigma\delta}$ -filter) if \mathcal{F} is an analytic (resp. F_σ -subset, $F_{\sigma\delta}$ -subset) of the power-set $\mathcal{P}(\omega) = 2^\omega$ endowed with the natural compact metrizable topology.

A filter \mathcal{F} is *measurable* (resp. *null*) if it is measurable (resp. has measure zero) with respect to the Haar measure on the Cantor cube 2^ω considered as the countable product of 2-element groups. It is well-known that a filter is measurable if and only if it is null. The relations between meager and null filters are not trivial and were investigated in [18] and [2]. Since each analytic filter is meager and null we get the following chain of properties of filters:

$$F_\sigma \Rightarrow \text{analytic} \Rightarrow \text{meager} \ \& \ \text{null}.$$

We are going to show that some meager and null filter \mathcal{F} admits an injective \mathcal{F} -convergent sequence in $\beta\omega$ while no F_σ -filter \mathcal{F} admits such a sequence. The latter fact holds more generally for analytic P^+ -filters.

A filter \mathcal{F} on ω is called a P -filter (resp. a P^+ -filter) if each countable subfamily $\mathcal{C} \subset \mathcal{F}$ has a pseudointersection A that belongs to \mathcal{F} (resp. to \mathcal{F}^+). Here

$$\mathcal{F}^+ = \{A \subset \omega : \forall F \in \mathcal{F} \ A \cap F \neq \emptyset\}$$

coincides with the union of all filters that contain \mathcal{F} . It is clear that each P -filter is a P^+ -filter. In particular, the Fréchet filter \mathcal{F} is both a P -filter and P^+ -filter.

For a filter \mathcal{F} on ω by $\chi(\mathcal{F})$ we denote its *character*. It is equal to the smallest cardinality $|\mathcal{B}|$ of the base $\mathcal{B} \subset \mathcal{F}$ that generates \mathcal{F} in the sense that $\mathcal{F} = \{F \subset \omega : \exists B \in \mathcal{B} \ B \subset F\}$. It is well-known that the character of each free ultrafilter on ω is uncountable. The uncountable cardinal $\mathfrak{u} = \min\{\chi(\mathcal{U}) : \mathcal{U} \in \beta\omega \setminus \omega\}$ is called the *ultrafilter number*, see [3], [19]. The *dominating number* \mathfrak{d} is the smallest cardinality $|D|$ of a cofinal subset D in the partially ordered set (ω^ω, \leq) , see [3], [19]. By Ketonen's Theorem [10], *each filter \mathcal{F} on ω with character $\chi(\mathcal{F}) < \mathfrak{d}$ is a P^+ -filter*.

Now we can establish some properties of filters \mathcal{F} admitting injective \mathcal{F} -convergent sequences in $\beta\omega$.

Theorem 4. *Assume that a filter \mathcal{F} admits an injective \mathcal{F} -convergent sequence $(x_n)_{n \in \omega}$ in $\beta\omega$.*

- (1) *If \mathcal{F} is a P^+ -filter, then for some set $A \in \mathcal{F}^+$ the filter $\mathcal{F}|A = \{F \cap A : F \in \mathcal{F}\}$ on A is an ultrafilter.*
- (2) $\chi(\mathcal{F}) \geq \min\{\mathfrak{d}, \mathfrak{u}\}$;
- (3) \mathcal{F} is not an analytic P^+ -filter;
- (4) \mathcal{F} is not an F_σ -filter.

Proof. 1. Assume that \mathcal{F} is a P^+ -filter. Let x_∞ be the \mathcal{F} -limit of the \mathcal{F} -convergent sequence $(x_n)_{n \in \omega}$ in $\beta\omega$. Since the sequence (x_n) is injective, there is $m \in \omega$ such that for every $n \geq m$ $x_n \neq x_\infty$ and hence we can fix a neighborhood U_n of x_∞ whose closure does not contain the point x_n . Since the sequence (x_k) \mathcal{F} -converges to x_∞ , for every $n \geq m$ the set $F_n = \{k \in \omega : x_k \in U_n\}$ belongs to the filter \mathcal{F} . Since \mathcal{F} is a P^+ -filter, the sequence $(F_n)_{n \geq m}$ has a pseudointersection $A \in \mathcal{F}^+$. It follows from the choice of the neighborhoods U_n that the set $\{x_n\}_{n \in A}$ is discrete in $\beta\omega$ and the sequence $(x_n)_{n \in A}$ is $\mathcal{F}|A$ -convergent to x_∞ . By Rudin's Theorem [16], the map $f : A \rightarrow \beta\omega$, $f : n \mapsto x_n$, has injective Stone-Ćech extension $\beta f : \beta A \rightarrow \beta\omega$, which implies that the filter $\mathcal{F}|A$ is an ultrafilter.

2. If $\chi(\mathcal{F}) < \min\{\mathfrak{d}, \mathfrak{u}\}$, then $\chi(\mathcal{F}) < \mathfrak{d}$ and by the Ketonen's Theorem [10] \mathcal{F} is a P^+ -filter. By the preceding statement, $\mathcal{F}|A$ is an ultrafilter for some set $A \in \mathcal{F}^+$. Consequently,

$$\mathfrak{u} \leq \chi(\mathcal{F}|A) \leq \chi(\mathcal{F}) < \mathfrak{u}$$

and this is a desired contradiction.

3. If \mathcal{F} is an analytic P^+ -filter, then by the first statement, $\mathcal{F}|A$ is an ultrafilter for some subset $A \in \mathcal{F}^+$. On the other hand, the filter $\mathcal{F}|A$ is analytic being a continuous image of the analytic filter \mathcal{F} . So, $\mathcal{F}|A$ cannot be an ultrafilter.

4. Assume that \mathcal{F} is an \mathcal{F}_σ -filter. In order to apply the preceding statement, it suffices to show that \mathcal{F} is a P^+ -filter. This is done in the following lemma. \square

Lemma 2. *Each F_σ -filter \mathcal{F} on ω is a P^+ -filter.*

Proof. According to a result of Mazur [12] (see also [17]), for the F_σ -filter \mathcal{F} there exists a lower semi-continuous submeasure ϕ on $\mathcal{P}(\omega)$ such that $\mathcal{F} = \{A \subset \omega : \phi(\omega \setminus A) < \infty\}$. Since $\mathcal{F} \neq \mathcal{P}(\omega)$, $\phi(\omega) = \infty$ and the subadditivity of ϕ implies that $\phi(F) = \infty$ for all $F \in \mathcal{F}$. It follows from $\mathcal{F} = \{A \subset \omega : \phi(\omega \setminus A) < \infty\}$ that a set $A \subset \omega$ belongs to \mathcal{F}^+ if and only if $\phi(A) = \infty$.

To show that \mathcal{F} is a P^+ -filter, fix any decreasing sequence of sets $(A_k)_{k \in \omega}$ in \mathcal{F} . Let $n_0 = 0$ and by induction construct an increasing sequence of positive integers $(n_k)_{k \in \omega}$ such that $\phi([n_k, n_{k+1}) \cap A_k) > k$ for every $k \in \omega$. Then the set $A = \bigcup_{k \in \omega} [n_k, n_{k+1}) \cap A_k$ is a pseudointersection of $(A_k)_{k \in \omega}$ and belongs to the family \mathcal{F}^+ as $\phi(A) = \infty$. \square

Let us remark that Lemma 2 cannot be generalized to $F_{\sigma\delta}$ -filters. The following example was suggested to the authors by Jonathan Verner.

Example 2. The filter $\mathfrak{F}r \otimes \mathfrak{F}r = \{A \subset \omega \times \omega : \{n \in \omega : \{m \in \omega : (n, m) \in A\} \in \mathfrak{F}r\} \in \mathfrak{F}r\}$ on $\omega \times \omega$ is an $F_{\sigma\delta}$ but not P^+ .

In light of Theorem 4 it is natural to ask the following

Question 1. *Does $\beta\omega$ contain an injective \mathcal{F} -convergent sequence for some analytic filter \mathcal{F} ?*

On the other hand, we have the following fact:

Theorem 5. *Each infinite compact Hausdorff space X contains an injective \mathcal{F} -convergent sequence for some meager and null filter \mathcal{F} .*

Proof. Choose any finite-to-one function $\xi : \omega \rightarrow \omega$ such that

$$\lim_{n \rightarrow \infty} |\xi^{-1}(n)| = \infty \quad \text{and} \quad \prod_{n \in \omega} (1 - 2^{-|\xi^{-1}(n)|}) = 0.$$

By Corollary 2, any infinite compact Hausdorff space X contains an injective \mathcal{F} -convergent sequence for some ξ -meager filter \mathcal{F} . It is clear that \mathcal{F} is meager. It remains to check that \mathcal{F} is null. The filter \mathcal{F} , being ξ -meager, lies in the union $\bigcup_{n \in \omega} \mathcal{F}_n$ where $\mathcal{F}_n = \{A \subset \omega : \forall k \geq n \ A \cap \xi^{-1}(k) \neq \emptyset\}$. It suffices to prove that each set \mathcal{F}_n has Haar measure zero. Observe that the set \mathcal{F}_n can be identified with the product $\prod_{k \geq n} (\mathcal{P}(\varphi^{-1}(k)) \setminus \{\emptyset\})$, which has Haar measure

$$\prod_{k \geq n} \frac{2^{|\varphi^{-1}(k)|} - 1}{2^{|\varphi^{-1}(k)|}} = \prod_{k \geq n} (1 - 2^{-|\varphi^{-1}(k)|}) = 0.$$

\square

Remark 1. After writing this paper the authors learned from V.Tkachuk that the meager property of the function space $C_p(\omega^*, 2)$ was also established by E.G. Pytkeev in his Dissertation [15, 3.24]. Game characterizations of topological spaces X with Baire function space $C_p(X, \mathbb{R})$ were given in [9] and [14].

1. ACKNOWLEDGMENTS

The authors would like to express their thanks to Alan Dow and Jonathan Verner for very stimulating discussions and to Vladimir Tkachuk for the information about Pytkeev's results on the Baire category of function spaces.

REFERENCES

- [1] A. Aviles, B. Cascales, V. Kadets, A. Leonov, *The Schur l_1 theorem for filters*, Zh. Mat. Fiz. Anal. Geom. **3**:4 (2007), 383–398.
- [2] T. Bartoszyński, M. Goldstern, H. Judah, S. Shelah, *All meager filters may be null*, Proc. Amer. Math. Soc. **117**:2 (1993), 515–521.
- [3] E. van Douwen, *The integers and topology*, Handbook of set-theoretic topology, 111–167, North-Holland, Amsterdam, 1984.
- [4] M. Ganichev, V. Kadets, *Filter convergence in Banach spaces and generalized bases*, in: General topology in Banach spaces (T.Banakh ed.), Nova Sci. Publ., Huntington, NY, 2001. p.61–69.
- [5] S. García-Ferreira, V. Malykhin, A. Tamariz-Mascarúa, *Solutions and problems on convergence structures to ultrafilters*, Questions Answers Gen. Topology **13**:2 (1995), 103–122.
- [6] R. Engelking, *General topology*, Heldermann Verlag, Berlin, 1989.
- [7] K.P. Hart, *Efimov's problem*, in: Open Problems in Topology II (E.Pearl ed.), Elsevier, 2007, P.171–177.
- [8] N. Lašnev, *On continuous decompositions and closed mappings of metric spaces*, Dokl. Akad. Nauk SSSR **165** (1965), 756–758 (in Russian).
- [9] D.J. Lutzer, R.A. McCoy, *Category in function spaces. I*, Pacific J. Math. **90**:1 (1980), 145–168.
- [10] J. Ketonen, *On the existence of P -points in the Stone-Čech compactification of integers*, Fund. Math. **92** (1976), no. 2, 91–94.
- [11] V.I. Malykhin, G. Tironi, *Weakly Fréchet-Urysohn and Pytkeev spaces*, Topology Appl. **104** (2000) 181–190.
- [12] K. Mazur, *F_σ -ideals and $\omega_1\omega_1^*$ -gaps in the Boolean algebras $P(\omega)/I$* , Fund. Math. **138**:2 (1991), 103–111.
- [13] V. Mykhaylyuk, *On questions connected with Talagrand's problem*, Mat. Stud. **29** (2008) 81–88.
- [14] E.G. Pytkeev, *The Baire property of spaces of continuous functions*, Mat. Zametki **38**:5 (1985), 726–740.
- [15] E.G. Pytkeev, *Spaces of continuous and Baire functions in weak topologies*, Doktor Sci. Dissertation, Ekaterinburg, 1993 (in Russian).
- [16] M.E. Rudin, *Types of ultrafilters*, 1966 Topology Seminar (Wisconsin, 1965) pp. 147–151 Ann. of Math. Studies, No. 60, Princeton Univ. Press, Princeton, N.J.
- [17] S. Solecki, *Analytic ideals*, Bull. Symbolic Logic **2**:3 (1996), 339–348.
- [18] M. Talagrand, *Compacts de fonctions mesurables et filtres non mesurables*, Studia Math. **67**:1 (1980), 13–43.
- [19] J. Vaughan, *Small uncountable cardinals and topology*, Open problems in topology, 195–218, North-Holland, Amsterdam, 1990.

(T.Banakh) IVAN FRANKO NATIONAL UNIVERSITY OF LVIV, UNIVERSYTETSKA 1, LVIV 79000, UKRAINE AND
 UNIWERSYTET HUMANISTYCZNO-PRZYRODNICZY JANA KOCHANOWSKIEGO, KIELCE, POLAND.

E-mail address: tbanakh@yahoo.com

URL: <http://www.franko.lviv.ua/faculty/mechmat/Departments/Topology/bancv.html>

(V.Mykhaylyuk) DEPARTMENT OF MATHEMATICS, YURIY FEDKOVYCH CHERNIVTSI NATIONAL UNIVERSITY, KOTSJUBYNSKOGO
 STR. 2, CHERNIVTSI 58012, UKRAINE.

E-mail address: vmykhaylyuk@ukr.net

(L.Zdomsky) KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC, UNIVERSITY OF VIENNA, WÄHRINGER STRASSE
 25, A-1090 WIEN, AUSTRIA.

E-mail address: lzdomsky@logic.univie.ac.at

URL: <http://www.logic.univie.ac.at/~lzdomsky/>